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The Hausdorf dimension of the Apollonian packing of circles

Peter B Thomas†‡ and Deepak Dhar§

Theoretical Physics Group, Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400 005, India

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Abstract. We formulate the problem of determining the Hausdorf dimension, d_f, of the Apollonian packing of circles as an eigenvalue problem of a linear integral equation. We show that solving a finite-dimensional approximation to this infinite-order matrix equation and extrapolating the results provides a fast algorithm for obtaining high-precision numerical estimates for d_f. We find that $d_f = 1.305\,686\,729\,(10)$. This is consistent with the rigorously known bounds on d_f, and improves the precision of the existing estimate by three orders of magnitude.

1. Introduction

The problem of tiling a plane with circular discs of variable radii is a very old one, and tradition associates the simplest of these with Apollonius of Perga in 200 BC. An illustration of the Apollonian packing is given in figure 1. One starts with three touching discs with arbitrary curvatures. Inside the resulting curvilinear triangle is inscribed a new circle, touching the other three. The process is repeated in each of the resulting curvilinear triangles *ad infinitum*. If the discs are *open*, that is, if they exclude all the points on their perimeters, then the *residual set* of points that do not fall within any of the discs form a fractal set that has been called the 'Apollonian gasket'.

This tiling has been popularized by Mandelbrot [1], and it has been suggested at various times as a model for mechanical gear-works, turbulence, textures of smectic liquid crystals and soap solutions, the motion of tectonic plates of the earth during earthquakes, etc. A somewhat recent review of the known results on the subject can be found in the book by Falconer [2]. The relation between the curvatures of mutually touching circles (or spheres) in two and three dimensions was first obtained by Söddy, who expressed it in a poem 'The Kiss Precise' [3]. A generalization to N dimensions was obtained by Gosset [4]. A very elegant proof of this generalization may be found in a recent paper by Söderberg [5].

A problem that has been of interest for the past several decades is that of determining the Hausdorf dimension d_f of the Apollonian gasket. It has been conjectured that, amongst the various different possible disc tilings of a plane, the Hausdorf dimension is the smallest for the Appolonian tiling [6]. One of the reasons for the enduring fascination of this problem has been its difficulty. It was first considered by Hirst [7] who showed that d_f was strictly

† Present address: Department of Physics, Brookhaven National Laboratory, Upton, NY 11973, USA.

‡ e-mail address: peter@tifrvax.bitnet

§ e-mail address: ddhar@tifrvax.bitnet

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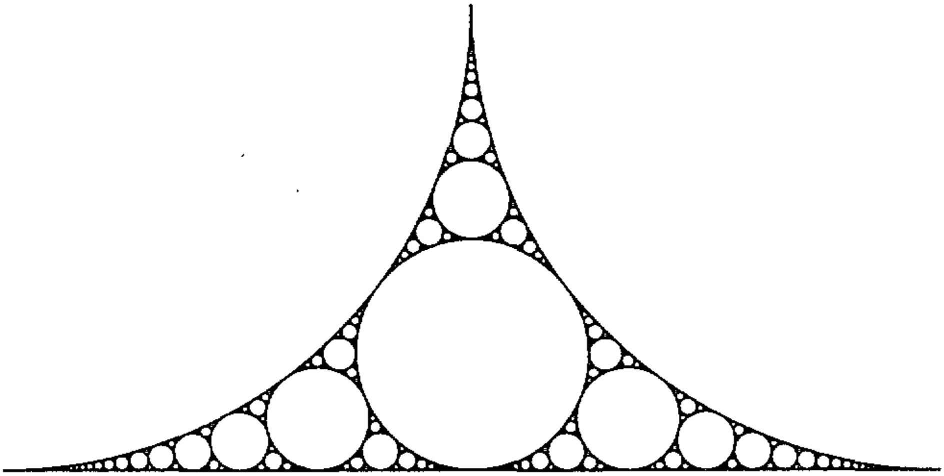

Figure 1. The Appolonian gasket that is generated from the curvilinear triangle bounded by two unit circles and a straight line (circle with zero curvature).

greater than 1, and had an upper bound of $\log 3/\log(1+2\sqrt{3})$. In a series of papers, Boyd [8, 9] found that

$$1.300\,197 < d_f < 1.314\,534 \tag{1.1}$$

which are still the best known rigorous bounds. A heuristic estimate obtained by Melzak was $d_f \approx 1.306\,951$ [10]. A recent numerical study by Manna and Hermann [11] gave the improved estimate

$$d_f = 1.305\,684 \pm 0.000\,010\,. \tag{1.2}$$

The problem has a rich mathematical structure and symmetry. The tiling can be generated by three specific Mobius transformations of the form $(\alpha z + \beta)/(\gamma z + \delta)$ in the complex plane. The three transformations can be expressed in terms of three complex 2×2 matrices formed out of α, β, γ and δ. These are generators of a discrete subgroup of $SL(2, C)$ which is, in fact, the symmetry group of a regular graph-like structure in a three-dimensional manifold of constant negative curvature [5]. Instead of using the three complex 2×2 matrices, one could use three integer 4×4 matrices [7]. In particular, this implies that if the curvatures of the three circles in the outer curvilinear triangle are 1, 1 and 0, then the curvatures of all the inscribed circles will also be integers. The problem of determining d_f *becomes equivalent to that of determining the properties of random products of non-*commuting matrices. Under a similarity transformation, these integer matrices transform into three special Lorentz boosts operating on light-like vectors, and they are therefore the generators of a discrete subgroup of the Lorentz group $SO(1, 3)$ [5]. For the Apollonian tiling on a sphere one has the same Lorentz boosts, but they operate on time-like vectors. The Apollonian tiling can thus be thought of as the problem of finding the distribution of velocities for a rocket which is given n impulses randomly, in three different directions by three jets fixed to the rocket.

Another reason why we chose to study this problem comes from the interesting structure of the functional equation it gives rise to. In the renormalization group approach to describing critical phenomena, one finds that physical quantities such as, say, the free energy, satisfy functional equations of the form

$$f(x') = c + \alpha f[R(x)] \tag{1.3}$$

where the components of x are the relevant couplings in the problem and $R(x)$ is the renormalization group transformation (in general nonlinear) of x under scale changes, and α is a positive constant. The unstable fixed points of (1.3) correspond to the critical points in the model, and one can study the singularities of f by linearizing the recurrence relations around these points. A simple generalization of (1.3) is the equation

$$f(x') = c + \alpha_1\, f[R_1(x)] + \alpha_2\, f[R_2(x)] \tag{1.4}$$

where R_1 and R_2 are different transformations of x, and α_1 and α_2 are positive constants. Such equations do not in general have fixed points, but much more complicated fractal attractors, and have not been subjected to much study. One expects these to be of possible relevance in systems subjected to two or more competing 'flows', as, for example, in turbulence. In this paper, we have formulated the problem of finding d_f in terms of solving a functional relation of this kind. In this problem, we have R_1, R_2 and R_3 as three *linear* transformations. However, even in this simple case we are not able to find d_f analytically. But we show that the numerical solution of a functional equation similar in structure to (1.4) provides an efficient algorithm to determine d_f. For example, using only moderate computing power we are able to improve the precision of the estimate of Manna and Hermann by three orders of magnitude.

The paper is organized as follows. In section 2 we formulate the problem in terms of solving an infinite system of coupled linear equations. Then in section 3 we discuss a similar but simpler problem that is exactly solvable, and show that it has an infinite number of solutions. We then discuss the additional conditions that have to be imposed on the solution to make it unique. In section 4 we describe our numerical algorithm, and present our results.

2. Formulation of the problem

Suppose a, b and c are the curvatures of the sides of a curvilinear triangle, and s is the curvature of the inscribed circle. The value of s can be obtained in terms of a, b and c by elementary geometry, and is given by the Söddy formula,

$$s = a + b + c + 2d \qquad \text{where} \qquad d \equiv \sqrt{ab + bc + ca}\,. \tag{2.1}$$

Equivalently, this can be written as

$$2(a^2 + b^2 + c^2 + s^2) = (s + a + b + c)^2\,. \tag{2.2}$$

If s' is the curvature of the circle inscribed by the curvilinear triangle formed by the circles a, b and s, we see that s' and c are the solutions of equation (2.2) treated as a quadratic in c for fixed a, b and s. This implies that $s' = 2a + 2b + 2s - c$. We represent the curvilinear triangle (a, b, c) by a 4-vector $\boldsymbol{X} = (a, b, c, s)$. If $\boldsymbol{X}_1'$ is the 4-vector corresponding to the curvilinear triangle bounded by (s, b, c), X_2' to the triangle bounded by (a, s, b) and X_3' to the triangle bounded by (a, b, s), then

$$X_i' = A_i X \qquad i = 1, 2, 3 \tag{2.3}$$

where $X^T = (a, b, c, s)$, and

$$A_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 2 & 2 & 2 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 2 & -1 & 2 & 2 \end{pmatrix} \qquad A_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 2 & -1 & 2 \end{pmatrix}. \tag{2.4}$$

After n levels of iteration, one has 3^n circles corresponding to the 3^n different products of the n matrices $A_{j_1}, A_{j_2}, \ldots, A_{j_n}$; $j_i = 1, 2, 3$. The matrices A_1, A_2 and A_3 are known as the Boyd matrices [8].

Let $\{s_i\}$, $i = 1$ to ∞, be the set of curvatures of all the inscribed discs in an Apollonian gasket that is bounded by three circles with curvatures a, b and c. We assume that these circles touch from the outside, so that a, b and c are non-negative. Let the s_i's be arranged in ascending order so that $s_1 \leqslant s_2 \leqslant s_3 \ldots$. The Melzak function is defined by

$$M(a, b, c; x) \equiv \sum_{i-1}^{\infty} s_i^{-x} \tag{2.5}$$

where x is some positive exponent. It is obvious that if $x = 2$ then $M(a, b, c; x)$ will be proportional to the area of the outermost curvilinear triangle, and if $x = 0$ then, since $\{s\}$ is an infinite set, M diverges. Hence there must be a critical value x_c such that M is finite for $x > x_c$ and infinite for $x \leqslant x_c$. It is easy to see that x_c is independent of a, b and c, and that in fact

$$x_c \equiv d_f \tag{2.6}$$

the Hausdorf dimension of the gasket [9]. Also, if $N(s)$ is the cumulative total of all circles having a curvature less than s, then $N(s) \sim s^{d_f}$ for large s. This fact was exploited by Melzak [10] and later, using faster computers, by Manna and Hermann [11]. Their estimates have been given in the introduction.

It is easy to see that the Melzak function satisfies the recursion relation

$$M(a, b, c; x) = s^{-x} + M(s, b, c; x) + M(a, s, c; x) + M(a, b, s; x) \tag{2.7}$$

where s is the curvature of the circle inside the curvilinear triangle bounded by three circles of curvatures a, b and c. M is a smooth, continuous, homogenous function of a, b and c,

$$M(\lambda a, \lambda b, \lambda c; x) = \lambda^{-x} M(a, b, c; x). \tag{2.8}$$

As $x \to x_c$, M diverges as $(x - x_c)^{-1}$. Near x_c, we can expand $M(a, b, c; x)$ in a power series in $(x - x_c)$,

$$M(a, b, c; x) = \frac{M_{-1}(a, b, c)}{x - x_c} + M_0(a, b, c) + \text{terms of order } (x - x_c). \tag{2.9}$$

Multiplying (2.9) by $(x - x_c)$ and taking the limit $x \to x_c$, we see that $M_{-1}(a, b, c)$ satisfies the homogenous equation

$$M_{-1}(a, b, c) = M_{-1}(s, b, c) + M_{-1}(a, s, c) + M_{-1}(a, b, s). \tag{2.10}$$

In our method, we numerically determine $M_{-1}(a, b, c)$, and also at the same time find x_c.

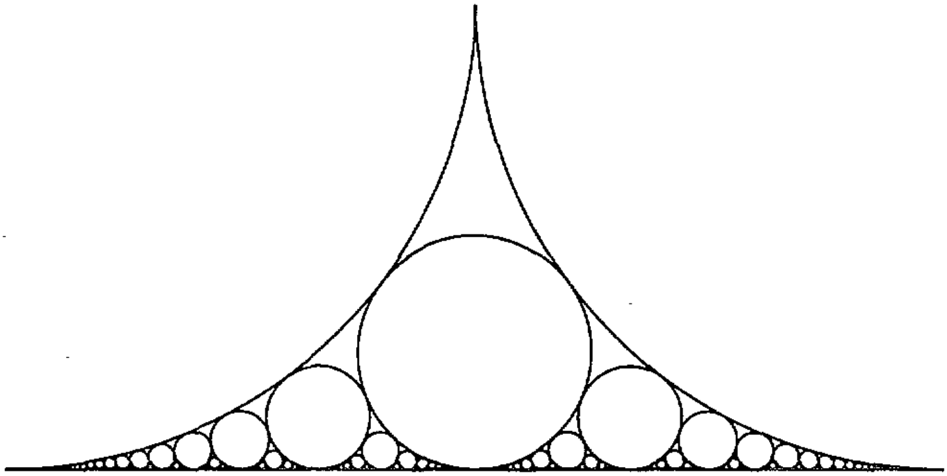

Figure 2. The subset of the Appolonian gasket which consists only of circles touching the straight line.

3. Boundary conditions

Equation (2.7) is of the form mentioned in (1.4), with three competing transformations, and does not have a simple fixed point. In fact, as we will proceed to show, there are an infinite number of solutions to the homogenous equation (2.10). Thus it needs to be supplemented with the correct boundary conditions, so that its solution is consistent with the definition (2.5). We first illustrate this by considering a simpler gasket, with similar properties.

3.1. An illustrative example

Consider a gasket that is generated by starting with a straight line and two mutually touching unit circles. An open circle is inscribed, touching all the sides of this curvilinear triangle. The process is repeated indefinitely for all the new curvilinear triangles, containing the straight line as one of its sides. The resulting set of points belonging to the circumference of at least one of these circles defines this gasket. Its construction is shown in figure 2. It happens that the Hausdorf dimension of this gasket has the trivial value unity. However, it is instructive to consider this case since the recursion relation analogous to (2.7) for this gasket has similar properties to (2.7) itself and can be solved analytically.

As for the Apollonian case, let $\{t_i\}$, $i = 1$ to ∞, be the set of curvatures of all the circles in this gasket arranged in ascending order, with a and b being the curvatures of its two outermost circles. Define

$$G(a, b; x) = \sum_{i=1}^{\infty} t_i^{-x} . \tag{3.1}$$

This definition of $G(a, b; x)$ implies the scaling property

$$G(\lambda a, \lambda b; x) = \lambda^{-x} G(a, b; x) . \tag{3.2}$$

At the nth iteration, there will be 2^n curvilinear triangles, each with a straight line as one of its sides. If a and b are the curvatures of the other two sides and t the curvature of the inscribed circle, then by putting $c = 0$ in the Söddy formula (2.1), we have

$$t = \left(\sqrt{a} + \sqrt{b}\right)^2 . \tag{3.3}$$

In order to find the distribution function $N(t)$, we define $\alpha = \sqrt{a}$ and $\beta = \sqrt{b}$. We now observe from (3.1) that $G(a, b; x)$ can be written as

$$G(\alpha^2, \beta^2; x) = \sum_{n_1, n_2} \frac{C(n_1, n_2)}{(n_1\alpha + n_2\beta)^{2x}} \tag{3.4}$$

where the summation runs over integers n_1 and n_2, $1 \leqslant n_1 \leqslant \infty$, $1 \leqslant n_2 \leqslant \infty$ and $C(n_1, n_2)$ is unity if n_1 and n_2 are *mutually coprime* and zero otherwise. Since for large n_1 and n_2, the mutually coprime pairs (n_1, n_2) are distributed roughly uniformly in the $(n_1\ n_2)$ plane with asymptotic density $6/\pi^2$ [12], we get

$$G(\alpha^2, \beta^2; x) \approx \frac{6}{\pi^2} \int \mathrm{d}n_1 \int \mathrm{d}n_2 \frac{1}{(n_1\alpha + n_2\beta)^{2x}} . \tag{3.5}$$

This integal converges for $x > 1$ and diverges for $x < 1$, which shows that the Hausdorf dimension of this gasket is unity. In fact, it follows from (3.5) that

$$G(\alpha^2, \beta^2; x) \to \frac{3}{\pi^2} \frac{(\alpha\beta)^{-1}}{(x-1)} \qquad \text{as } x \to 1^+ . \tag{3.6}$$

It is clear from (3.1) that $G(a, b; x)$ satisfies the recursion relation

$$G(a, b; x) = \left(\sqrt{a} + \sqrt{b}\right)^{-2x} + G(t, b; x) + G(a, t; x) . \tag{3.7}$$

$G(a, b; x)$ has a first-order pole at $x = 1$ and can be written in the form

$$G(a, b; x) = \frac{G_{-1}(a, b)}{(x-1)} + G_0(a, b) + \cdots . \tag{3.8}$$

From (3.8) we find that $G_{-1}(a, b)$ satisfies the homogenous equation

$$G_{-1}(a, b) = G_{-1}(t, b) + G_{-1}(a, t) . \tag{3.9}$$

Due to the two parameters a and b, we note that the recursion relations (3.7) or (3.9) do not have a simple fixed point.

One can construct other solutions to this functional equation. For example, one finds by inspection that (3.7) has a solution for all x

$$G'(a, b; x) = -\frac{1}{2}\left(\frac{1}{a^x} + \frac{1}{b^x}\right) . \tag{3.10}$$

This implies that if G is defined by (3.4) then for all positive k, $(1 + k)G - kG'$ is positive and satisfies (3.7). We have thus constructed an infinite number of solutions to (3.7), and it needs to be supplemented with boundary conditions to select the correct solution defined by (3.4).

In order to find this boundary condition, we note that, for all x, $G(1, 0; x)$ is infinite. This is because the set $\{t\}$ contains an infinite number of circles of unit radius, between two parallel straight lines. It is easy to see from (3.7) that

$$G(1, \epsilon; x) = \left(1 + \sqrt{\epsilon}\right)^{-2x} + G\left[\left(1 + \sqrt{\epsilon}\right)^2, \epsilon; x\right] + G\left[1, \left(1 + \sqrt{\epsilon}\right)^2; x\right] . \tag{3.11}$$

Combining (3.11) with the scaling property (3.2), we find that $G(1, \epsilon; x)$ must diverge as $1/\sqrt{\epsilon}$, for small ϵ.

3.2. Boundary conditions for the recursion relations of the Melzak function

As in the above example, one can also construct spurious solutions to the recursion relation (2.7). For example, an obvious choice is to join the vertices of the centres of the circles. For any given curvilinear triangle, the area of the corresponding triangle formed from the centres then satisfies the homogenous equation

$$A(a,b,c) = A(s,b,c) + A(a,s,c) + A(a,b,s) \tag{3.12}$$

where $A(a,b,c) = \sqrt{ab+bc+ca}/abc$. It is obvious that this is a spurious solution since, for example, it diverges when $a = 1$, $b = 1$ and $c = 0$, while the $M_{-1}(a,b,c)$ defined in (2.9) is finite. Further, it satisfies the homogenity property (2.8) only for $x = 2$.

Therefore, as in the previous example, it is also important to obtain the supplementing boundary conditions for the Apollonian case. We note that $M(1,0,0;x)$ corresponds to adding xth powers of curvatures of circles starting from two parallel lines and a unit circle. It is therefore clear that the ratio $M(1,0,0;x)/M(1,1,0;x)$ is infinite for all values of x. Arguments as in the previous section show that $M(1,\epsilon,0;x)/M(1,1,0;x)$ diverges as $1/\sqrt{\epsilon}$ for all x. Keeping the symmetry of M under permutations of a, b and c in mind, we see that $M(a,b,c;x)$ diverges as $1/d$ for all x, when d is small. Using (2.8) we find that the leading behaviour for small d is

$$M(a,b,c;x) \sim \frac{1}{d(a+b+c)^{x-1}}\,. \tag{3.13}$$

It is also easy to see from (2.5) and (2.8) that

$$(2+\varepsilon+\delta)^x\, M(1,1+\varepsilon,\delta;x) \sim 2^x\, M(1,1,0;x)\left[1+\mathrm{O}(\varepsilon^2)+\mathrm{O}(\delta)\right] \tag{3.14}$$

from which we find, by making an expansion around $M(1,0,0;x)$, that at $x = x_c$,

$$\frac{1}{M_{-1}(a,b,c)} \sim Kd(a+b+c)^{x_c-1}\left[1-\frac{abc}{(2x_c+1)(a+b+c)d^2}+\mathrm{O}(d^2)\right] \tag{3.15}$$

where K is a constant independent of a, b and c. One is therefore motivated to define a new function $F_x(a,b,c)$ by the equation

$$M(a,b,c;x) = \frac{d(a+b+c)^{2-x}}{(2x+1)(a+b+c)d^2-abc}F_x(a,b,c)\,. \tag{3.16}$$

Since the leading divergence of M is taken care of by the prefactor, the function F is finite for all $a,b,c > 0$. In fact, our numerical results show that F is a very slowly varying function, with maximum variation $\sim 0.4\%$ for $x = x_c$. To leading order in d we can write

$$F_x(a,b,c) \simeq F_x(1,0,0)\left[1+\frac{ud^2}{(a+b+c)^2}\right] \tag{3.17}$$

where u is a function of x, but independent of a, b and c. $F_x(1,0,0)$ is well behaved, and one finds by substituting (3.17) in (2.7) that in the limit $(a,b,c) \to (1,0,0)$

$$F_x(1,0,0) = \frac{1}{2x-1} + \frac{2^{2-x}}{2x-1}\,F_x(1,1,0)\,. \tag{3.18}$$

It is clear from (2.8) that

$$F_x(\lambda a,\lambda b,\lambda c) = F_x(a,b,c)\,. \tag{3.19}$$

Therefore we can assume, without loss of generality, that $a+b+c = 1$. If the initial choices of a, b and c are positive, then the relevent section of the plane is the equilateral triangle in the first octant, with vertices $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$. As is the case for the Melzak function, $F_x(a,b,c)$ is also invariant under permutations of a, b and c. It can be well approximated by extrapolation from its values at a few points.

4. Numerical algorithm for the estimation of x_c

In this section, we develop a systematic approximation scheme in terms of the levels of iteration of the recursion relation (2.7) to get high precision estimates for d_f.

We rewrite (2.7) in terms of $F_x(a,b,c)$

$$F_x(a,b,c) = C_x(a,b,c) + L_x(a,b,c)\,F_x(s,b,c) + L_x(b,c,a)\,F_x(a,s,c) + L_x(c,a,b)\,F_x(a,b,s) \tag{4.1}$$

where s is the curvature of the inner circle, given by the Söddy formula (2.1), and

$$C_x(a,b,c) = \frac{(2x+1)(a+b+c)(ab+bc+ca) - abc}{s^x(a+b+c)^{2-x}(ab+bc+ca)^{1/2}}$$

$$L_x(a,b,c) = \left(\frac{s+b+c}{a+b+c}\right)^{2-x}\left[\frac{sb+bc+cs}{ab+bc+ca}\right]^{1/2} \times \left[\frac{(2x+1)(a+b+c)(ab+bc+ca)-abc}{(2x+1)(s+b+c)(sb+bc+cs)-sbc}\right]. \tag{4.2}$$

We note that if we start with positive a, b and c, then under the recursion the three new triplets are also all positive. Hence the recursion is closed on the equilateral triangle in the positive octant of the plane $a+b+c=1$. In fact, if (4.1) is iterated repeatedly, then the resulting rescaled vectors (a,b,c) have a fractal attractor on the triangle. The points $(1,0,0)$ and $(\frac{1}{2},\frac{1}{2},0)$ (using (3.19) we can refer to this as the $(1,1,0)$ point, and shall do so hereafter) lie on the attractor. The total attractor can be obtained recursively by starting from $(a,b,c)=(1,0,0)$ and its permutations, and iterating it indefinitely. The part of the attractor after nine levels of recursion is shown in figure 3.

As the functional equation (4.1) relates the values of F on the attractor only to other points on the attractor, it is sufficient to solve the recursion relations for points only lying on the attractor by using (3.18) as the first equation, and then using (4.1) from the $(1,1,0)$ point onwards. These form an infinite system of linear simultaneous equations, which can conveniently be expressed in matrix form,

$$(\mathcal{I} - \mathcal{L})F = C \tag{4.3}$$

where $\mathcal{I}$ is the identity matrix and $\mathcal{L}$ is an infinite-dimensional square matrix. It is clear that $\mathcal{L}$ is an extremely sparse upper-triangular matrix, with three (in general) non-zero entries in each row. The formal solution of the simultaneous equations is

$$F = (\mathcal{I} - \mathcal{L})^{-1}C\,. \tag{4.4}$$

By definition, $F_x(a,b,c)$ diverges at $x = x_c$, and hence $(\mathcal{I}-\mathcal{L})$ is singular at $x = x_c$. We note that the norm of $\mathcal{L}$ decreases with increasing x. Since for $x > x_c$, $F_x(a,b,c)$ is well behaved, it follows that *the maximum eigenvalue of $\mathcal{L}$ is unity at $x = x_c$*†.

Approximations to x_c can be obtained by truncating this matrix equation to n levels of iteration of the recursion relations. If we then use the permutation symmetry, there

† Although the matrix $\mathcal{L}$ is upper-triangular with all its diagonal elements zero, it can have non-trivial eigenvalues, since it is an infinite-dimensional matrix.

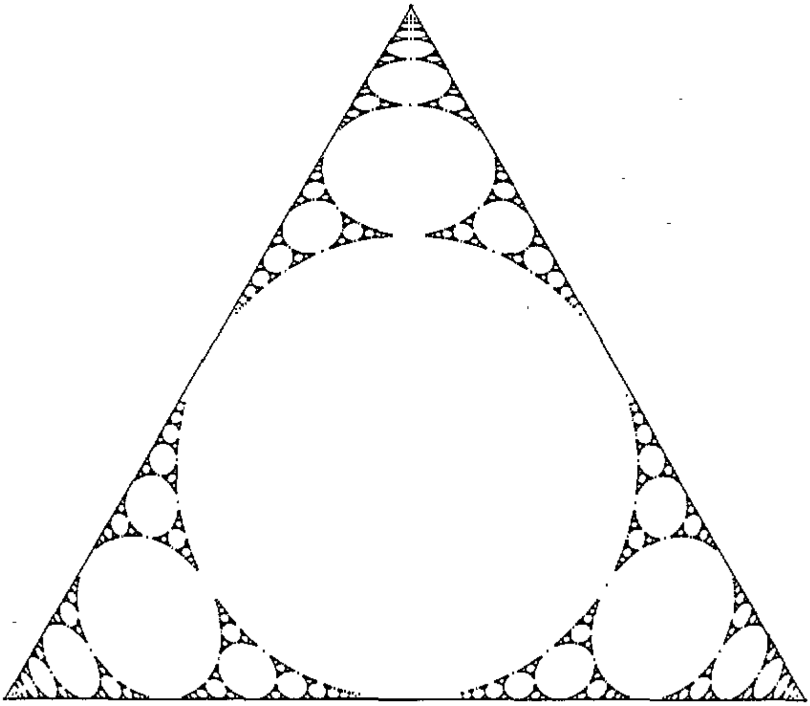

Figure 3. The fractal attractor of the recursion relation for the Melzak function is generated by starting from the point (1, 0, 0) and its permutations. This figure has been generated after nine levels of recursion.

are $(3^n + 1)/2$ new equations which do not close. We close the equations by using the smoothness of F and interpolating its value at the unknown points from its values at neighbouring points. We illustrate this procedure for $n = 0$ and $n = 1$ below.

As a first approximation ($n = 0$), we assume that $F_x(1, 0, 0) \approx F_x(1, 1, 0)$. Since $F_x(1, 0, 0)$ and $F_x(1, 1, 0)$ diverge at $x = x_c$, we find, from (3.18),

$$\frac{2^{2-x_c}}{2x_c - 1} \simeq 1\,. \tag{4.5}$$

Solving this transcendental equation, we get the estimate

$$x_c \simeq 1.307\,85 \tag{4.6}$$

which is reasonably close to the correct value $x_c = 1.3057$ given in [11].

For $n = 1$, we use the equation for $F_x(1, 1, 0)$,

$$F_x(1, 1, 0) = C_x(1, 1, 0) + 2L_x(1, 1, 0)\, F_x(4, 1, 0) + L_x(0, 1, 1)\, F_x(4, 1, 1)\,. \tag{4.7}$$

This, along with the equation for $F_x(1, 0, 0)$ (equation (3.18)), forms two linear equations for $F_x(1, 0, 0)$ and $F_x(1, 1, 0)$. However, they are not closed, since they also involve the unknowns $F_x(4,1,0)$ and $F_x(4,1,1)$. We close the equations by taking into account the leading variation of $F_x(a,b,c)$ on its arguments (3.17). From this we find that

$$F_x(4,1,0) \simeq \tfrac{9}{25}F_x(1,0,0) + \tfrac{16}{25}F_x(1,1,0) \qquad F_x(4,1,1) \simeq F_x(1,1,0)\,. \tag{4.8}$$

The equation is now expressed completely in terms of $F_x(1,0,0)$ and $F_x(1,1,0)$. Combining (4.8) with (3.18), we have two linear simultaneous equations

$$\begin{aligned} F_x(1,0,0) - \frac{2^{2-x}}{2x-1}F_x(1,1,0) &= \frac{1}{2x-1} - \frac{18}{25}L_x(1,1,0)F_x(1,0,0) \\ &\quad + \left[1 - \frac{32}{25}L_x(1,1,0) - L_x(0,1,1)\right]F_x(1,1,0) \simeq C_x(1,1,0)\,. \end{aligned} \tag{4.9}$$

The solution of (4.9) for x_c is determined by the transcendental equation

$$\frac{5^{1-x_c}}{25}\left[\frac{18}{2x_c-1} + 2^{3+x_c}\right] + \frac{(2x_c+1)3^{3-x_c}}{27(2x_c+1)-2} \simeq 1 \tag{4.10}$$

which gives

$$x_c \simeq 1.305\,74\,. \tag{4.11}$$

This is correct to four decimal places [11].

For $n \geqslant 2$, we interpolate the value of F at any of the unknown points (a,b,c) from its values at the three nearest points (a_1,b_1,c_1), (a_2,b_2,c_2) and (a_3,b_3,c_3) that enclose it. This is done by writing, for each new point (a,b,c),

$$F_x(a,b,c) = \gamma_1 F_x(a_1,b_1,c_1) + \gamma_2 F_x(a_2,b_2,c_2) + \gamma_3 F_x(a_3,b_3,c_3)\,. \tag{4.12}$$

Of the various interpolation formulae for γ_1, γ_2 and γ_3 that we tried, we found empirically that the following approximation has good convergence in n:

$$F_x(a,b,c) \simeq p + q\,d^2 + r\,(abc)\,. \tag{4.13}$$

The constants p, q and r are allowed to be different for different values of (a,b,c). They are determined for the unknown points in terms of the values of F at the points (a_1,b_1,c_1), (a_2,b_2,c_2) and (a_3,b_3,c_3). We note that this interpolation formula respects the symmetry of $F_x(a,b,c)$ under the permutations of its arguments.

The size of the truncated matrix at the nth level is

$$\mathrm{Size}(n) = \frac{3^n + 2n + 3}{4}\,. \tag{4.14}$$

The resulting finite approximation to $\mathcal{L}$ (which we denote by $\mathcal{L}_n$), is no longer upper-triangular, and therefore has non-trivial eigenvalues. We have determined the value of $x(n)$ when the maximum eigenvalue of $\mathcal{L}_n$ is unity, for $n = 2,3,\ldots,11$. We found that simple iteration of an arbitrary vector under repeated multiplication by $\mathcal{L}_n$ converges very quickly. This also enabled us to exploit the sparseness of the matrix $\mathcal{L}_n$. The entire data was computed in a few hours CPU time by a Silicon Graphics Iris 4D/310S machine. We have extrapolated the solutions to $n = \infty$ from $x(n)$, assuming exponential convergence, by the formula

$$x^*(n) \approx x(n) - \frac{[x(n) - x(n-1)]^2}{x(n) - 2x(n-1) + x(n-2)} \tag{4.15}$$

by sequentially choosing three values of $x(n)$ at a time. We have also calculated $x^\dagger(n)$ from $x^*(n)$ by assuming power-law convergence,

$$x^\dagger(n) = x^*(n) - \frac{k}{n^6}. \tag{4.16}$$

The constant k was obtained by taking two sequential values of $x^*(n)$ at a time. The results are shown in table 1. All computations have been done in double precision, and the numbers are correct to all the decimal places quoted in the table. The data for $n = 0$ and $n = 1$, have been computed explicitly earlier in this section ((4.6) and (4.11)). However, these have not been included since for $n = 0$ no interpolation was necessary, and for $n = 1$ we used two-point interpolation (4.8), while for $n \geqslant 2$ three-point interpolation (obtained from (4.13)) was used.

Table 1. Numerical results and extrapolation to $n = \infty$.

n	Size(n)	$x(n)$	$x^*(n)$	$x^\dagger(n)$
2	4	1.305 720 399 77	—	—
3	9	1.305 698 983 71	—	—
4	23	1.305 690 777 96	1.305 68 568 085	—
5	64	1.305 688 087 01	1.305 686 773 95	1.305 687 162 3
6	186	1.305 687 215 18	1.305 686 797 35	1.305 686 809 1
7	551	1.305 686 921 54	1.305 686 772 41	1.305 686 756 0
8	1 645	1.305 686 813 07	1.305 686 749 53	1.305 686 730 9
9	4 926	1.305 686 769 29	1.305 686 739 65	1.305 686 730 0
10	14 768	1.305 686 749 75	1.305 686 734 01	1.305 686 727 6
11	44 293	1.305 686 740 31	1.305 686 731 50	1.305 686 728 2

A reasonable estimate from the data is that the Hausdorf dimension of the Apollonian gasket is

$$d_f \simeq 1.305\,686\,729\,(10). \tag{4.17}$$

This is consistent with Boyd's rigorous bounds [9], and is also in good agreement with the numerical estimate of Manna and Hermann [1].

To summarize, we have made an improved numerical estimate of the Hausdorf dimension d_f of the Apollonian gasket by formulating the problem in terms of determining the value of a smooth function $F_x(a, b, c)$ on a fractal set in an equilateral triangle. We have used a finite set of linear relations relating the value of the function at some points on the triangle, and have used interpolation to approximate its value at others. Analytical determination of d_f remains an open problem.

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